Newton polygons for A_inf

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Newton Polygons for A_{inf}

Recall (last time)

Let K be a non-arch. field, $v : K \to \mathbb{R} \cup \{\infty\}$

Let $f \in K[T]$ be a polynomial.

$\mathrm{Newt}_{poly}(f) :=$ lower convex hull of $\{(i, v(a_i))\}_{i=0}^{n}$

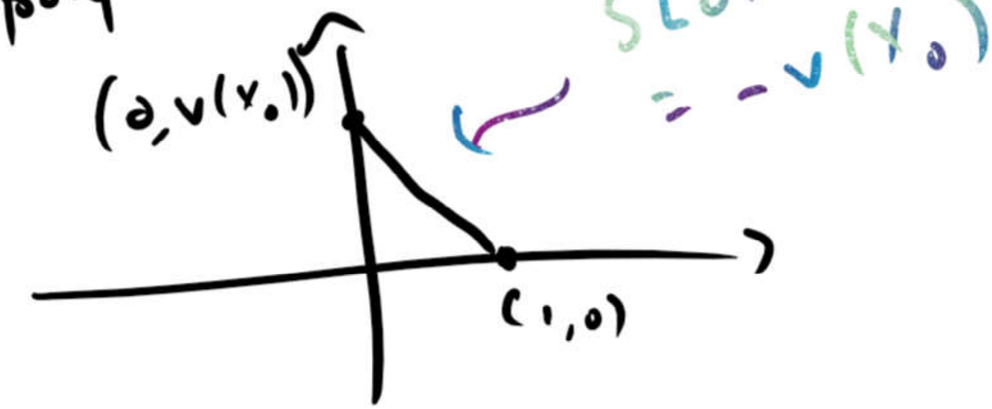
where $f = a_0 + a_1 T + \dots + a_n T^n$

Thm Let $x_0, \dots, x_n \in \bar{K}$ be the roots of f. Then $-v(x_0), \dots, -v(x_n)$ are the slopes of $\mathrm{Newt}_{poly}(f)$.

Example if $x_0 \in K$,

$\mathrm{Newt}_{poly}(T - x_0) =$ SLOPE

$\text{Newt}_{\text{poly}}(1 - x_0) =$



If we can prove

$$\text{Newt}_{\text{poly}}(f \cdot g) = \text{Newt}_{\text{poly}}(f) \star \text{Newt}_{\text{poly}}(g)$$

then we have proven the thm

$\star$: convolution of PL functions.

<u>Idea</u> : use $\mathcal{L}$, "Legendre transform"

Start with a fn ← $\mathbb{R} \cup \{\pm\infty\}$

$$\varphi \in \text{Maps}(\mathbb{R}, \overline{\mathbb{R}}) =: \mathcal{F}$$

$$\mathcal{L} : \mathcal{F} \longrightarrow \mathcal{F}$$

$$\mathcal{L}(\varphi)\,(\lambda) = \inf_{x \in \mathbb{R}} \{\varphi(x) + \lambda x\}$$

Keys A • $\tilde{\mathcal{L}} \circ \mathcal{L}(\varphi)$ is the (non-extendable) convex hull of φ

B • $\mathcal{L}, \tilde{\mathcal{L}}$ are inverse transforms on continuous PL functions.

C • $\mathcal{L}(\varphi \star \psi) = \mathcal{L}(\varphi) + \mathcal{L}(\psi)$

$$A \Rightarrow \mathcal{L}(\mathrm{Newt}_{\mathrm{poly}}(f))(r) = \inf_{i \in \mathbb{Z}} \{ v(a_i) + ri \} =: v_r(f)$$

$$v_r(f) = \inf \left\{ v(f(x)) \;\middle|\; \begin{array}{l} x \in \overline{K} \text{ with} \\ v(x) \leq r \end{array} \right\}$$

i.e., inf on a circle in $\overline{K}$

Lemma v_r is a valuation

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Power Series

- Can do NP for $f \in \mathcal{O}_K[[T]]$

$\mathrm{Newt}(f)$ = decreasing, lower convex hull of $\{(i, v(a_i))\}_{i \geq 0}$

Thm (Lazard)

If $\lambda \neq 0$ is a slope of $\mathrm{Newt}(f)$,

$\Rightarrow \exists\, \alpha \in \hat{\overline{K}}$ w/

- $f(\alpha) = 0$
- $v(\alpha) = -\lambda$

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Goal:

- Develop a thy of NPs for $f \in A_{\inf}$
- Prove an analogy of Lazard's thm.

Move on to $A_{\inf}$

Recall: $E/\mathbb{Q}_p$ finite, π uniformizer of $\mathcal{O}_E$

$\mathcal{O}_E/\pi \simeq \mathbb{F}_q$,

$F/\mathbb{F}_q$ is alg closed, non-arch. extension

$v: F \longrightarrow \mathbb{R} \cup \{\infty\}$

(Back of Head: $E = \mathbb{Q}_p$, $F = \widehat{\overline{\mathbb{F}_p((T))}}$)

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$A_{inf} := W_{\mathcal{O}_E}(\mathcal{O}_F)$ $\left(= W\left(\widehat{\overline{\mathbb{F}_p[[T]]}}\right)\right)$

$\mathcal{O}_F$ perfect $\mathbb{F}_p$-algebra, so $\exists!$
"Teichmüller expansion"

$f \in A_{inf}$, $f = \sum_{i \geq 0} [a_i] \pi^i$, where

$$[\cdot] : \mathcal{O}_F \longrightarrow W_{\mathcal{O}_E}(\mathcal{O}_F)$$

Reminder: addition + mult. are complicated.

Guess: $\text{Newt}(f)$: decreasing, lower convex hull of $\{(i, v(a_i))\}_{i \geq 0}$

... to be worth

For this definition to be worth anything, we should have:

$$\mathrm{Newt}(fg) = \mathrm{Newt}(f) \star \mathrm{Newt}(g)$$

$$f, g \in A_{\inf}$$

For $r \geq 0$, $f = \sum_{i \geq 0} [a_i] \pi^i \in A_{\inf}$,

Set $\displaystyle v_r(f) := \inf_{i \in \mathbb{N}} \{ v(a_i) + ri \}$

<u>Note</u> : $v_r(f)$ is not necessarily attained for $r = 0$

but is attained for $r > 0$.

It follows from facts about $\mathcal{L}$, that $\mathrm{Newt}(f) =$ convex, decreasing, PL s.t.

$p^{\_}$ s.t.

$$\mathcal{L}(\mathrm{Newt}(f)) = \begin{cases} v_r(f), & r \geq 0 \\ -\infty, & r < 0 \end{cases}$$

To prove that

$$\mathrm{Newt}(fg) = \mathrm{Newt}(f) \star \mathrm{Newt}(g),$$

it suffices to prove

$$\star \quad v_r(fg) = v_r(f) + v_r(g)$$

To prove this, I introduce a family of norms "Gauss norms".

Def $\rho \in (0,1)$, $|f|_\rho := \sup_{i \in \mathbb{N}} |a_i| \rho^i$

Note: if $\rho = q^{-r}$, $|\cdot|_\rho = q^{-v_r(\cdot)}$

Lemme (1.4.2 in F-F)

Fix $k > 0$ integer, $f \in \sum [a_i] \pi^i \in A_{inf}$

Fix $k \geq 0$ integer, $f \in \sum_{i \geq 0} [a_i] \pi^i \in A_{inf}$

$$N_k(f) = \sup_{0 \leq i \leq k} |a_i| .$$

Then:

① If $r \in [0,1] \cap |F|$, then $N_k(f) \leq r$ iff, for $a \in O_F$ with $|a| = r$, we have

$$f \in A_{inf}[a] + A_{inf}\pi^{k+1}$$

(pf: mult. of Teichmüller + uniqueness of " ")

③ $N_k(f+g) \leq \sup\{N_k(f), N_k(g)\}$

④ $N_k(fg) \leq \sup_{m+n=k} \{N_m(f) N_n(g)\}$

⑤ $|f| = \sup N_k(f) \rho^k$

⑤ $|f|_\rho = \sup_{k \geq 0} N_k(f)\rho$

Pf ① ✓

③ $f = \sum_{i \geq 0} [a_i]\pi^i \quad g = \sum_{j \geq 0} [b_j]\pi^j$

Coeffs of $f+g$: polynomials in coeffs of f & g.

$\underline{A} = (a_0, \ldots, a_k) \subseteq \mathcal{O}_F$

$\underline{B} = (b_0, \ldots, b_k) \subseteq \mathcal{O}_F$

$(a_i, b_j)_{i,j=0}^{k} \subseteq \mathcal{O}_F$

$\Rightarrow (\underline{A}, \underline{B}) = (c)$

Moreover, $|C| = \sup_{0 \leq i,j \leq k} (|a_i|, b_j)$

$\leq (N_k(f), N_k(g))$

$$= \sup\left(N_k(f), N_k(g)\right)$$

$$f+g \in A_{inf}[c] + A_{inf}\bar{\pi}^{k+1}$$

$$\Rightarrow N_k(f+g) \leq \sup\{N_k(f), N_k(g)\}$$

(5) $$\sup_{k\geq 0} N_k(f)\rho^k = \sup_{k\geq 0} \sup_{0\leq i\leq k} |a_i|\rho^k$$

$$= \sup_{i\geq 0} \sup_{k\geq i} |a_i|\rho^k$$

$(\rho \in (0,1))$ $$= \sup_{i\geq 0} |a_i|\rho^i$$

$\square$

upshot $\Rightarrow$

$$|f+g|_\rho \leq \sup\{|f|_\rho, |g|_\rho\}$$

$|fg|_p = \sup \ldots$

$$|fg|_p \le |f|_p \, |g|_p$$

$\Rightarrow$ $|\cdot|_p$ is sub-multiplicative

$=$

<u>Recall</u> $f, g \in A_{\inf}$, want to prove

$$\mathrm{Newt}(fg) = \mathrm{Newt}(f) \star \mathrm{Newt}(g)$$

$\rightsquigarrow$ reduce to $v_r(fg) = v_r(f) + v_r(g)$ for $r \in [0, \infty)$

$\Longleftrightarrow$ (cont.) $|fg|_p = |f|_p \, |g|_p$ for $p \in (0,1)$

Key idea of pf of multiplicativity of $|\cdot|_p$:

$f = \sum [a_i] \pi^i$, can

$f = \sum [a_i] \pi^i$, can

find the smallest n w/

$$|f|_\rho = |a_n| \rho^n$$

write:

$$f = f' + [a_n]\pi^n + \pi^{n+1} f'',$$

where $|f'|_\rho < |a_n| \rho^n$

Do the same with g.

Multiply out and sub-mult

that we just proved

$$=$$

$\square$

$$\|f\|_\rho := \sup_{i \in \mathbb{N}} |a_i| \rho^i \qquad \rho \in (0,1)$$

(alt: read Lecture 5 of Lurie, "Norms")

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<u>Goals</u>

① NPs for $f \in A_{inf}$ ✓

② Prove analog of Lazard's thm

<u>Thm</u> (Analog of Lazard)

Let $f \in A_{inf}$, $\lambda \neq 0$ slope of $\mathrm{Newt}(f)$. Then $\exists\, a \in \mathcal{O}_F$ w/ $v(a) = -\lambda$, so that

$$f = (\pi - [a])\, g$$

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$g \in A_{inf}$

Such a are not unique

P involves $|Y|$

$$|Y|_{[0,\infty)} = \{ I \subseteq A_{inf} \mid I \text{ is generated by a distinguished elt} \}$$

$$= \mathrm{Prim}' / A_{inf}^{\times}$$

$$\mathrm{Prim}' = \{ \textstyle\sum [a_i]\pi^i \text{ where } |a_0| < 1,\ |a_1| = 1 \}$$

"0" $\longleftrightarrow$ $\{(\pi)\}$

"∂" $\longleftrightarrow$ $\{(\pi)\}$

$$|Y| = |Y|_{[0,\infty)} \setminus \{(\pi)\}$$

$$|Y| \longleftrightarrow \left\{ (C, \iota) \;\middle|\; \begin{array}{l} C/E \text{ complete, non-arch extension,} \\ \iota : C^\flat \xrightarrow{\sim} F \end{array} \right\}$$

= "space of char 0 untilts of F"

Notation $y \in |Y|$

- $\mathfrak{p}_y \subseteq A_{inf}$ is the ideal
- ξ_y is a distinguished generator of $\mathfrak{p}_y$
- $C_y := (A_{inf}/\mathfrak{p}_y)[\frac{1}{\pi}]$

- $\Theta_y : A_{inf} \longrightarrow C_y$

(In scheme thy, $f \in R$
Spec R.
$f(p) \in K_p$)

- $v_y : C_y \longrightarrow \mathbb{R} \cup \{\infty\}$ is the valuation on C_y, characterized by:

$$\Theta_y \; A_{inf} \to C_y$$
$$\uparrow [\cdot]$$
$$\mathcal{O}_F$$

$$v_y(\Theta_y([a])) = v(a)$$

- $f \in A_{inf}$, $f(y) := \Theta_y(f)$
$$\uparrow$$
$$C_y$$

$$v(f(y)) = v_y(f(y))$$

$$v(f(y)) = v_y(f(y))$$

<u>Claim</u> $\exists$ a notion on $|Y|$ that will make it a complete metric space.

<u>Def</u> If $y_1, y_2 \in |Y|$

$$d(y_1, y_2) := V_{Y_1}\left(\Theta_{y_1}\left(\xi_{y_2}\right)\right)$$

$\xi_{y_2} \in A_{inj}$, $\Theta_{y_1}(\xi_{y_2}) \in C_{y_1}$, $V_{Y_1}(\Theta_{y_1}(\xi_{y_2})) \in \mathbb{R}$

<u>Rmk</u> NOT obvious this is symmetric!

this is symmetric.

$$\cdot\ d(y_1, 0) := v(\underbrace{\pi(y_1)}_{\pi \in A_{inf} \longrightarrow \mathcal{O}_{y_1}})$$

<u>Miracle</u> $d(-,-)$ is an ultrametric valuation, i.e.,

① $d(y_1, y_2) = d(y_2, y_1)$

② $d(y_1, y_3) \geq \inf\left\{ \begin{matrix} d(y_1, y_2), \\ d(y_2, y_3) \end{matrix} \right\}$

③ $d(y_1, y_2) = \infty \Leftrightarrow y_1 = y_2$

<u>Def</u> For $r \geq 0$, set

$$\underline{a}_r := \{ f \in A_{inf} \mid V_0(f) \geq r \}$$

$$= \{ \sum [a_i] \pi^i \mid v(a_i) \geq r \text{ for all } i \}$$

$$= \left\{ \sum [a_i]\pi^i \;\middle|\; v(a_i) \geq r \text{ for all } i \right\}$$

This is an ideal of A_{inf}

Then

$$d(y_1, y_2) = \sup_{r \geq 0} \left\{ P_{y_1} + \underline{a}_r = P_{y_2} + \underline{a}_r \right\}$$

ideal in A_{inf} ideal in A_{inf}

Take this as a given.

$$d(y_1, y_2) = \infty \iff$$

$$P_{y_1} + \underline{a}_r = P_{y_2} + \underline{a}_r \text{ for all } r \geq 0$$

but $$A_{inf} = \lim_{r \geq 0} A_{inf} / \underline{a}_r$$

$\Rightarrow \quad P_{y_1} = P_{y_2}$

$\Leftrightarrow \quad y_1 = y_2$

Prop For $r \in (0, \infty)$,

$(|Y_r|, d)$ is complete,

where $|Y_r| := \{ y \in |Y| \mid d(y, 0) = r \}$

$= \left\{ \begin{matrix} \text{untilts } (C, \iota) \\ \text{of } F, \text{ with} \\ C/E \end{matrix} \;\middle|\; v_C(\pi) = r \right\}$

Indication of pf

$\{y_n\}$ is a Cauchy sequence in $|Y_r|$,

goal is to prove it converges in $|Y_r|$.

Claim 1 for any $r' > 0$, the

ideals $\{ P_n + a_{r'} \}_n$ are

ideals $\{ \mathfrak{p}_{y_n} + \mathfrak{a}_{r'} \}_n$ are

constant for $n \gg 0$

(follows from $d(,)$ + Cauchy)

<u>Claim 3</u>

$I_{r'} := (\mathfrak{p}_{y_n} + \mathfrak{a}_{r'}) / \mathfrak{a}_{r'} \subseteq A_{inf} / \mathfrak{a}_{r'}$

constant for $n \gg 0$

<u>Trick 3</u> Let r' vary,

$\varprojlim I_{r'} \subseteq \varprojlim A_{inf} / \mathfrak{a}_{r'}$

$= A_{inf}$

Set $I := \varprojlim I_{r'}$

Have to prove it is principal, gen by distinguished.

Thm Let $f \in A_{inf}$, $\lambda \neq 0$ a slope of $\mathrm{Newt}(f)$. Then $\exists\, a \in \mathcal{O}_F$ w/ $v(a) = -\lambda$, so that

$$f = (\pi - [a])\, g$$

in A_{inf}.

"Pf":

- reduce to primitive elements
- construct a Cauchy sequence in $|Y_{-\lambda}|$; this converges by above
- $\pi - [a]$ will be a distinguished generator of this limiting ideal.